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Blending Fourier and Chebyshev Interpolation*

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Over the rectangle $\Omega = (-1, 1) \times (-\pi, \pi)$ of $\mathbf{R}^2$, interpolation involving algebraic polynomials of degree M in the x direction, and trigonometric polynomials of degree N in the y direction is analyzed. The interpolation nodes are Cartesian products of the Chebyshev points $x_j = \cos \pi j/M$, $j = 0,..., M$, and the equispaced points $y_l = (l/N - 1)\pi$, $l = 0,..., 2N - 1$. This interpolation process is the basis of those spectral collocation methods using Fourier and Chebyshev expansions at the same time. For the convergence analysis of these methods, an estimate of the L^2-norm of the interpolation error is needed. In this paper, it is shown that this error decays like $N^{-r} + M^{-s}$ provided the interpolation function belongs to the non-isotropic Sobolev space $H^{r,s}(\Omega)$. © 1987 Academic Press, Inc.

1. INTRODUCTION AND BASIC NOTATIONS

Several numerical approximations of partial differential equations using spectral methods give a solution which is a finite expansion in terms of trigonometric (Fourier) polynomials in some directions and of Chebyshev polynomials in the others. This is for instance the case of those problems which are set in simply shaped domains, whose solution is periodic in some directions, and submitted to Dirichlet or Neumann boundary conditions in the remaining directions. A remarkable example is represented by the Taylor-Couette flow problem (see, e.g., [8, 10, 11]).

We will consider here a 2-dimensional domain $\Omega = (-1, 1) \times (-\pi, \pi)$ though the results that will be proved can be extended to any domain of the form $(-1, 1)^m \times (-\pi, \pi)^n$, $n, m \geqslant 1$. The stability and convergence analysis of the Chebyshev–Fourier spectral method relies upon the estimate of the interpolation error. The interpolation nodes (those where the differential equation is collocated) are the product of the Gauss-Chebyshev points in the interval $(-1, 1)$, and of a set of equispaced points in $(-\pi, \pi)$.

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This choice allows one to get the most of accuracy from the numerical method, and to use the fast Fourier transform to carry out the computations (see [5]).

In this paper we estimate the L^2-norm of the interpolation error. It is shown that this error vanishes as $M^{-r}+N^{-s}$, where M and N are the degree of the approximation in the x and y direction, respectively, provided the function to be interpolated belongs to the non-isotropic Sobolev space $H^{r,s}(\Omega)$ (see, e.g., [7]). Thus, the function is allowed to have different regularities in the different directions. This estimate is optimal, for the exponents of M^{-1} and N^{-1} are the highest possible, and no assumption is made about the ratio M/N. Therefore, M and N are not asked to vanish at the same rate.

The leading idea of this paper is to carry out the proof on the auxiliary domain $\Omega^0=(-\pi M, \pi M)\times(-\pi N, \pi N)$. In Ω^0, trigonometric polynomials of degree M in x and N in y, undergo to Bernstein-type inequalities whose constants are independent of either M and N.

The above idea was used first in [12] to carry out the error analysis for Fourier interpolation in one space variable, and then in [3] for the combined Fourier and finite element interpolation.

We denote with $\Theta=(a, b)$ an open interval of $\mathbf{R}$, and with $C_p^\infty(\Theta)$ the set of restrictions to Θ of the infinitely differentiable functions of $\mathbf{R}$ which are periodic of period $b-a$. The Sobolev space $H_p^s(\Theta)$, for integer $s\geqslant 0$, is the closure of $C_p^\infty(\Theta)$ with respect to the norm

$$\|u\|_{s,\Theta}=\left(\sum_{\alpha=0}^{s}\|D^\alpha u\|^2_{L_2(\Theta)}\right)^{1/2}.$$

If s is not an integer, then the completion is made with respect to the norm

$$\|u\|_{s,\Theta}=(\|u\|^2_{[s],\Theta}+|u|^2_{s,\Theta})^{1/2}$$

where $[s]=s-\sigma$ is the integral part of s, $0<\alpha<1$, and

$$|u|_{s,\Theta}=\left(\int_\Theta\int_\Theta\frac{|D^{[s]}(u(x)-u(y))|^2}{|x-y|^{1+2\sigma}}\,dx\,dy\right)^{1/2}$$

is the seminorm of orders s of u.

To introduce the Fourier interpolation we set for any integer $N>0$

$$S_N=\operatorname{span}\{e^{iky},\ -N\leqslant k\leqslant N-1\}.$$

For any function u, continuous on $[-\pi, \pi]$, let $F_N u\in S_N$ be its (Fourier) interpolant at the points

$$y_l=\left(\frac{l}{N}-1\right)\pi,\qquad 0\leqslant l\leqslant 2N-1. \tag{1.1}$$

Then if $u \in H_p^s(-\pi, \pi)$ for some $s > \frac{1}{2}$ we have (see [3, Lemma 1.9])

$$\|u - F_N u\|_{\nu,(-\pi,\pi)} \leqslant C N^{\nu - s} |u|_{s,(-\pi,\pi)}, \qquad 0 \leqslant \nu \leqslant s. \tag{1.2}$$

Let now $w(x) = (1 - x^2)^{-1/2}$, $-1 < x < 1$, be the Chebyshev weight function. We denote with $L_w^2(-1, 1)$ the space of functions whose square is integrable for the measure $w(x)\,dx$. For any positive integer r, $H_w^r(-1, 1)$ denotes the weighted Sobolev space of those functions whose derivatives of order up to r belong to $L_w^2(-1, 1)$. If r is a positive real number then $H_w^r(-1, 1)$ is defined by complex interpolation (see, e.g., [1, Chap. 4; 6]).

Following [9, 7], for any positive real numbers r, s we define

$$H^{r,s}(\Omega) = L_y^2(H_x^r) \cap H_y^s(L_x^2)$$

where

$$L_y^2(H_x^r) = \left\{ u\colon (-\pi, \pi) \to H_w^r(-1, 1) \mid \|u\|_{L_y^2(H_x^r)}^2 = \int_{-\pi}^{\pi} \|u(y)\|_{H_w^r(-1,1)}^2 dy < \infty \right\}.$$

Moreover, for any integer s,

$$H_y^s(L_x^2) = \left\{ u\colon (-\pi, \pi) \to L_w^2(-1, 1) \mid \|u\|_{H_y^s(L_x^2)}^2 = \sum_{j=0}^{s} \int_{-\pi}^{\pi} \|D^j u(y)\|_{L_w^2(-1,1)}^2 dy < \infty \right\};$$

while if s is real this space is defined by complex interpolation.

The space $H^{r,s}(\Omega)$ is a Hilbert space with the norm

$$\|u\|_{r,s,\Omega} = (\|u\|_{L_y^2(H_x^r)}^2 + \|u\|_{H_y^s(L_x^2)}^2)^{1/2}.$$

Finally $H_p^{r,s}(\Omega)$ will denote the closure with respect to the norm $\|r\|_{r,s,\Omega}$ of $C_p^\infty(\Omega)$ (the space of restrictions to Ω of the infinitely differentiable functions, periodic with period 2π along the y direction).

2. The Interpolation Process and the Domain Ω^0

We introduce first the Chebyshev interpolation in the interval $\{-1 \leqslant x \leqslant 1\}$. Let us define the set of points:

$$\vartheta_j = \left(\frac{j}{M} - 1\right)\pi, \qquad j = 0, \ldots, 2M - 1; \tag{2.1}$$

$$x_j = \cos \vartheta_j, \qquad j = 0, \ldots, M. \tag{2.2}$$

The latter are the nodes of the Chebyshev–Gauss–Lobatto formula (e.g., [4]); they are symmetrically distributed around the point $x=0$.

Let $\mathbf{P}_M$ denote the space of algebraic polynomials of the variable x of degree $\leqslant M$, and, for any $u \in C^0[-1, 1]$, let $C_M u \in \mathbf{P}_M$ be the interpolant of u at the points (2.2). Furthermore we set

$$u^*(\vartheta) = u(\cos \vartheta), \qquad -\pi \leqslant \vartheta \leqslant \pi. \tag{2.3}$$

We note that

$$\int_{-1}^{1} u(x)\, w(x)\, dx = \frac{1}{2} \int_{-\pi}^{\pi} u^*(\vartheta)\, d\vartheta. \tag{2.4}$$

Moreover, for any $r \geqslant 0$ we have

$$\|v^*\|_{r,(-\pi,\pi)} \leqslant \|v\|_{H^r_w(-1,1)}, \qquad \forall r \geqslant 0. \tag{2.5}$$

We recall that the Chebyshev interpolant has the form

$$C_M u(x) = \sum_{k=0}^{M} \tilde{u}_k T_k(x), \qquad T_k(x) = \cos(k\vartheta),$$

where $\vartheta = \arccos x$ and T_k is the Chebyshev polynomial of degree k. The $\tilde{u}_k$'s are the discrete Chebyshev coefficients of u, i.e., $\tilde{u}_k = (\pi/M) \sum^*{}_{j=0}^{M} u(x_j)\, T_k(x_j)$ (the asterisk means that the first and last term of the sum must be halved). From (2.2), (2.3) and the parity of the function $(C_M u)^*$ it follows that

$$(C_M u)^*(\vartheta_j) = u^*(\vartheta_j), \qquad j = 0,..., 2M-1.$$

Therefore, since

$$(C_M u)^*(\vartheta) = \sum_{k=-M}^{M} u_k^* \exp(ij\vartheta), \qquad u_k^* = \tfrac{1}{2}\tilde{u}_k,$$

we conclude that $(C_M u)^*$ is the unique function of the space

$$S_M^* = \left\{ v \colon v(\vartheta) = \sum_{k=-M}^{M} \alpha_k \exp(ik\vartheta),\ \alpha_M = \alpha_{-M} \right\}$$

which interpolates the function u^* at the points (2.1). Then (see [2, Theorem 1.1])

$$\|u^* - (C_M u)^*\|_{\mu,(-\pi,\pi)} \leqslant C M^{\mu - r} |u^*|_{r,(-\pi,\pi)}, \qquad 0 \leqslant \mu \leqslant r,\ r > \tfrac{1}{2}. \tag{2.6}$$

We define now two auxiliary domains (see Fig. 1)

$$\Omega^* = \{(\vartheta, y) \colon -\pi < \vartheta < \pi,\ -\pi < y < \pi\},$$

$$\Omega^0 = \{(\xi, \eta) \colon -M\pi < \xi < M\pi,\ -N\pi < \eta < N\pi\},$$

and two mappings

$$\Phi \colon \Omega^0 \to \Omega^*, \qquad \Phi(\xi, \eta) = (\xi/M, \eta/N),$$

$$\Psi \colon \Omega^* \to \Omega, \qquad \Psi(\vartheta, y) = (\cos\vartheta, y).$$

Then for any function $v \colon \Omega \to \mathbf{C}$ we define $v^* = v \circ \Psi$ and $v^0 = v^* \circ \Phi (= v \circ \Psi \circ \Phi)$. The following relation can be easily checked

$$|v^*(\cdot, y)|_{r,(-\pi,\pi)} = M^{r-1/2} |v^0(\cdot, \eta)|_{r,(-M\pi,M\pi)}, \qquad r \geqslant 0. \tag{2.7}$$

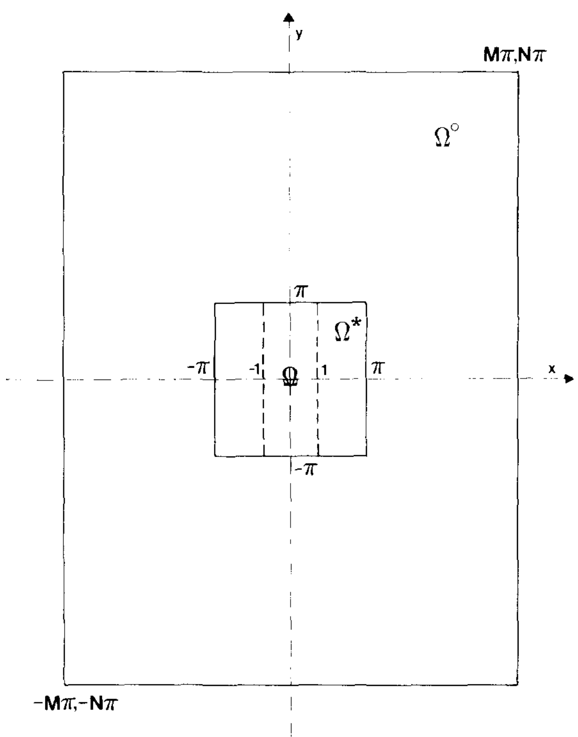

FIG. 1. The domains Ω, Ω^*, and Ω^0.

Similarly we have

$$|v^*(\vartheta,\cdot)|_{s,(-\pi,\pi)}=N^{s-1/2}|v^0(\xi,\cdot)|_{s,(-N\pi,N\pi)},\qquad s\geqslant 0. \tag{2.8}$$

We introduce now the finite dimensional space $V_{M,N}=\mathbf{P}_M\otimes S_N$ and the Chebysev/Fourier interpolation operator $I_{M,N}\colon C^0(\bar\Omega)\to V_{M,N}$ such that

$$I_{M,N}u(x_j,y_l)=u(x_j,y_l),\qquad j=0,...,M,\ l=0,...,2N-1, \tag{2.9}$$

The points x_j and y_l were defined by (2.2) and (1.1), respectively. For any M, N, $I_{M,N}u$ is uniquely defined, and

$$I_{M,N}u=(C_M\circ F_N)\,u=(F_N\circ C_M)\,u.$$

As we shall see, this operator induces an interpolation operator $I^0_{M,N}$ on the master domain Ω^0. For this, we define

$$V^0_{M,N}=\{v^0=v\circ\Psi\circ\Phi,\ v\in V_{M,N}\}.$$

Let us set

$$S^{*\,0}_M=\{v^0=v^*\circ\Phi_1,\ v^*\in S^*_M\}$$

and

$$S^0_N=\{v^0=v\circ\Phi_2,\ v\in S_N\}.$$

The scalar functions Φ_1 and Φ_2 are the components of the mapping Φ, namely $\Phi_1(\xi)=\xi/M$ and $\Phi_2(\eta)=\eta/N$. It follows that

$$V^0_{M,N}=S^{*\,0}_M\otimes S^0_N.$$

For any function $z\in C^0(\bar\Omega^0)$, we denote with $I^0_{M,N}z\in V^0_{M,N}$ the interpolant of z at the points (x^0_j,y^0_l), $j=0,...,2M-1$, $l=0,...,2N-1$, where $x^0_j=\Phi_1^{-1}(\vartheta_j)=M\vartheta_j$, and $y^0_l=\Phi_2^{-1}(y_l)=Ny_l$. Then we have

$$(I_{M,N}u)^0=I^0_{M,N}u^0\qquad\text{for all } u\in C^0(\bar\Omega). \tag{2.10}$$

In the sequel the symbol $\mathscr{L}(X;Y)$ will denote the space of linear and continuous functionals from the Hilbert space X with values into the Hilbert space Y. Moreover, let $H^{r,s}_p(\Omega^0)$, be the space which is formally defined as $H^{r,s}_p(\Omega)$, provided in that definition the space H^r_x is replaced with $H^r_p(-\pi M,\pi M)$, and the space H^s_y is replaced with $H^s_p(-\pi N,\pi N)$.

LEMMA 2.1. *For any couple (r,s) of real numbers satisfying $r^{-1}+s^{-1}<2$, there is a constant C^0 independent of both M and N such that*

$$\|I^0_{M,N}\|_{\mathscr{L}(H^{r,s}_p(\Omega^0);L^2(\Omega^0))}\leqslant C^0. \tag{2.11}$$

Proof. Let us set $\chi_{j,l}(\xi, \eta) = \exp(ij\xi + il\eta)$. It is not hard to see that if z is any continuous functions of $\bar{\Omega}^0$, then it is interpolant has the form

$$I^0_{M,N} z(\xi, \eta) = \sum_{j=-M}^{M}{}^{*} \sum_{l=-N}^{N-1} (z, \chi_{j,l})_{M,N} \, \chi_{j,l}(\xi, \eta)$$

where

$$(z, v)_{M,N} = \pi^2 \sum_{j=0}^{2M-1} \sum_{l=0}^{2N-1} z(x^0_j, y^0_l) \, v(x^0_j, y^0_l).$$

We recall that (see, e.g., [2])

$$(z, v)_{M,N} = \int_{\Omega^0} z\bar{v} \, d\xi \, d\eta \qquad \text{if both } z, v \text{ belong to } V^0_{M,N}.$$

Therefore

$$\|I^0_{M,N} z\|^2_{L^2(\Omega^0)} = (I^0_{M,N} z, I^0_{M,N} z)_{M,N} = \pi^2 \sum_{j=0}^{2M-1} \sum_{l=0}^{2N-1} |z(x^0_j, y^0_l)|^2.$$

If we define $\Omega^0_{j,l} = (x^0_j, x^0_{j+1}) \times (y^0_l, y^0_{l+1})$ then $\overline{\Omega^0} = \bigcup \{ \overline{\Omega^0_{j,l}}, j = 0, ..., 2M-1, l = 0, ..., 2N-1 \}$ (see Fig. 2).

Moreover, as $1/r + 1/s < 2$, then $H^{r,s}(\Omega^0_{j,l}) \subset C^0(\overline{\Omega^0_{j,l}})$ (see, e.g., [3, Lemma 1.3]). Therefore,

$$\|I^0_{M,N} z\|_{L^2(\Omega^0)} \leqslant \pi^2 \sum_{j=0}^{2M-1} \sum_{l=0}^{2N-1} C_{j,l} \|z\|_{H^{r,s}(\Omega^0_{j,l})}.$$

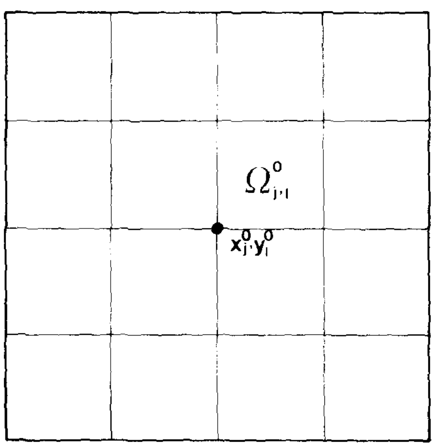

FIG. 2. The decomposition of Ω^0.

The constant $C_{j,l}$ depends on r, s and on the measure of $\Omega^0_{j,l}$ $(=\pi^2)$, thus we can set $\tilde{C}=C_{j,l}$ and this constant is independent of M, N, j and l. Now we observe that

$$\sum_{j=0}^{2M-1}\sum_{l=0}^{2N-1}\|z\|^2_{r,s,\Omega^0_{j,l}}\leqslant\|z\|^2_{H^{r,s}_p(\Omega^0)}$$

whence (2.11) holds taking $C^0=\tilde{C}\pi^2$.

3. Error Estimate in $L^2(\Omega)$

In this section we give an estimate of the L^2-norm of the interpolation error $u-I_{M,N}u$ for any function $u\in C^0(\bar{\Omega})$. To this end, we note that

$$\|u-I_{M,N}u\|^2_{0,0,\Omega}=\int_{-\pi}^{\pi}dy\int_{-1}^{1}|u-I_{M,N}u|^2\,w(x)\,dx$$

$$=\frac{1}{2}\int_{-\pi}^{\pi}dy\int_{-\pi}^{\pi}|u^*-(I_{M,N}u)^*|^2\,d\vartheta \qquad \text{(by 2.4))}$$

$$=\frac{1}{2}\frac{1}{MN}\|u^0-I^0_{M,N}u^0\|_{L^2(\Omega^0)} \qquad \text{(by (2.7), (2.8), (2.10))}.$$

If we denote by E the identity operator, then, obviously, $(I^0_{M,N}-E)\,z=0$ for all $z\in V^0_{M,N}$. Then $u^0-I^0_{M,N}u^0=(E-I^0_{M,N})(u^0-z)$ for all $z\in V^0_{M,N}$. It follows that

$$\|u-I_{M,N}u\|^2_{0,0,\Omega}\leqslant\frac{1}{2}\frac{1}{MN}\|E-I^0_{M,N}\|^2_{\mathscr{L}(H^{r,s}_p(\Omega^0);\,L^2(\Omega^0))}\times\inf_{z\in V^0_{M,N}}\|u^0-z\|^2_{H^{r,s}_p(\Omega^0)}.$$

Using the result of Lemma 2.1 and the triangle inequality it follows that

$$\|u-I_{M,N}u\|_{0,0,\Omega}\leqslant\sqrt{1+(C^0)^2}\,\frac{1}{\sqrt{MN}}\inf_{z\in V^0_{M,N}}\|u^0-z\|_{H^{r,s}_p(\Omega^0)} \quad \text{if } r^{-1}+s^{-1}<2. \tag{3.1}$$

An estimate of the infimum on the right-hand side of (3.1) is now needed. Let $P_{y,N}$ denote the orthogonal projection operator from $L^2(-\pi,\pi)$ onto S_N. Then (see [12; 3, Lemma 1.7])

$$\|v-P_{y,N}v\|_{\nu,(-\pi,\pi)}\leqslant CN^{\nu-\sigma}|v|_{\sigma,(-\pi,\pi)}, \qquad 0\leqslant\nu\leqslant\sigma, \tag{3.2}$$

provided $v \in H_p^\sigma(-\pi, \pi)$. Similarly, if $P_{x,M}$ is the orthogonal projection operator from $L^2(-\pi, \pi)$ onto S_M^*, then

$$\|v - P_{x,M} v\|_{\mu,(-\pi,\pi)} \leqslant CM^{\mu-\rho} |v|_{\rho,(-\pi,\pi)}, \qquad 0 \leqslant \mu \leqslant \rho, \tag{3.3}$$

provided $v \in H_p^\rho(-\pi, \pi)$.

If now $P_{y,N}^0$ denotes the orthogonal projection operator from $L^2(-\pi N, \pi N)$ upon S_N^0, and $P_{x,M}^0$ that from $L^2(-\pi M, \pi M)$ upon $S_M^{*\,0}$, then

$$P_{x,M}^0 z^0 = (P_{x,M} z)^0, \qquad P_{y,N}^0 v^0 = (P_{y,N} v)^0,$$

for all z and v in $L^2(-\pi, \pi)$. From the above relations and from (2.7), (2.8), (3.2), and (3.3) we obtain

$$\|z^0 - P_{x,M}^0 z^0\|_{r,(-\pi M,\pi M)} \leqslant C(r) |z^0|_{r,(-\pi M,\pi M)}, \qquad \forall z^0 \in H_p^r(-\pi M, \pi M),\ r \geqslant 0, \tag{3.4}$$

$$\|v^0 - P_{y,N}^0 v^0\|_{s,(-\pi N,\pi N)} \leqslant C(s) |v^0|_{s,(-\pi N,\pi N)} \qquad \forall v^0 \in H_p^s(-\pi N, \pi N),\ s \geqslant 0. \tag{3.5}$$

We are now going to establish a Bramble–Hilbert type lemma for trigonometric approximation in Ω^0. This result will then be used to get the error bound for the interpolation error.

LEMMA 3.1. *Let $u^0 \in H_p^{r,s}(\Omega^0)$ for some $r \geqslant 0$, $s \geqslant 0$. Then there exists a constant C^{00}, depending on r, s but independent of N and M such that*

$$\inf_{z \in V_{M,N}^0} \|u^0 - z\|_{H_p^{r,s}(\Omega^0)} \leqslant C^{00} \left\{ \int_{-\pi M}^{\pi M} d\xi |u^0|_{s,(-\pi N,\pi N)}^2 + \int_{-\pi N}^{\pi N} d\eta |u^0|_{r,(-\pi M,\pi M)}^2 \right\}. \tag{3.6}$$

Proof. For any function z of the space S_N^0 we have

$$\|z\|_{s,(-\pi N,\pi N)} \leqslant C \|z\|_{L^2(-\pi N,\pi N)}, \qquad \forall s \geqslant 0, \tag{3.7}$$

where C is a constant independent of N (see [3, Lemma 2.1]). Similarly, if $z \in S_M^{*\,0}$, then there is a constant C independent of M such that

$$\|z\|_{r,(-\pi M,\pi M)} \leqslant C \|z\|_{L^2(-\pi M,\pi M)}, \qquad \forall r \geqslant 0. \tag{3.8}$$

We have

$$\|u^0 - z\|_{H^{r,s}(\Omega^0)} = \int_{-\pi N}^{\pi N} d\eta \|u^0 - z\|_{r,(-\pi M,\pi M)}^2 + \int_{-\pi M}^{\pi M} d\xi \|u^0 - z\|_{s,(-\pi N,\pi N)}^2. \tag{3.9}$$

Taking $z = P^0_{x,M} P^0_{y,N} u^0$ and noting that $z = P^0_{x,M} u^0 - P^0_{x,M}(u^0 - P^0_{y,N} u^0)$ we get

$$\int_{-\pi N}^{\pi N} d\eta \| u^0 - z \|^2_{r,(-\pi M,\pi M)}$$

$$\leqslant 2 \left\{ \int_{-\pi N}^{\pi N} d\eta \| u^0 - P^0_{x,M} u^0 \|^0_{r,(-\pi M,\pi M)} \right.$$

$$\left. + \int_{-\pi N}^{\pi N} d\eta \| P^0_{x,M}(u^0 - P^0_{y,N} u^0) \|^2_{r,(-\pi M,\pi M)} \right\}$$

$$\leqslant C \left\{ \int_{-\pi N}^{\pi N} d\eta | u^0 |^2_{r,(-\pi M,\pi M)} + \int_{-\pi N}^{\pi N} d\eta \| P^0_{x,M}(u^0 - P^0_{y,N} u^0) \|^2_{0,(-\pi M,\pi M)} \right\}$$

(by (3.4) and (3.8)).

Now we note that

$$\int_{-\pi N}^{\pi N} d\eta \| P^0_{x,M}(u^0 - P^0_{y,N} u^0) \|^2_{0,(-\pi M,\pi M)} \leqslant \int_{-\pi M}^{\pi M} d\xi \| u^0 - P^0_{y,N} u^0 \|^2_{0,(-\pi N,\pi N)}.$$

Then, using (3.5) we conclude that

$$\int_{-\pi N}^{\pi N} d\eta \| u^0 - z \|^2_{r,(-\pi M,\pi M)}$$

$$\leqslant C \left\{ \int_{-\pi N}^{\pi N} d\eta | u^0 |^2_{r,(-\pi M,\pi M)} + \int_{-\pi M}^{\pi M} d\xi | u^0 |^2_{s,(-\pi N,\pi N)} \right\}. \tag{3.10}$$

Similarly, noting that $z = P^0_{y,N} P^0_{x,M} u^0 = P^0_{y,N} u^0 - P^0_{y,N}(u^0 - P^0_{x,M} u^0)$, and using (3.4), (3.5), (3.7), we conclude that

$$\int_{-\pi M}^{\pi M} d\xi \| u^0 - z \|^2_{s,(-\pi N,\pi N)} \leqslant C \left\{ \int_{-\pi M}^{\pi M} d\xi | u^0 |^2_{s,(-\pi N,\pi N)} \right.$$

$$\left. + \int_{-\pi N}^{\pi N} d\eta | u^0 |^2_{r,(-\pi M,\pi M)} \right\}. \tag{3.11}$$

Now (3.6) is a consequence of (3.9), (3.10), and (3.11). ∎

We can finally state the main result of this section.

THEOREM 3.1. *For any couple of positive real numbers r, s such that $r^{-1}+s^{-1}<2$ and any $u \in H_p^{r,s}(\Omega)$, we have*

$$\|u-I_{M,N}u\|_{0,0,\Omega} \leqslant C(M^{-r}+N^{-s})\|u\|_{r,s,\Omega} \tag{3.12}$$

where C is a positive constant independent of both N and M.

Proof. From (2.8), (2.3) and (2.4) we get

$$\int_{-\pi M}^{\pi M} d\xi |u^0|^2_{s,(-\pi N,\pi N)} = N^{1-2s} \int_{-\pi M}^{\pi M} d\xi |u^*|^2_{s,(-\pi,\pi)}$$

$$= 2(MN)\, N^{-2s} \int_{-1}^{1} w(x)\, dx |u|^2_{s,(-\pi,\pi)}. \tag{3.13}$$

Furthermore, from (2.7), (2.3) and (2.4) it follows

$$\int_{-\pi N}^{\pi N} d\eta |u^0|^2_{r,(-\pi M,\pi M)} = M^{1-2r} \int_{-\pi N}^{\pi N} d\eta |u^*|^2_{r,(-\pi,\pi)}$$

$$\leqslant (MN)\, M^{-2r} \int_{-\pi}^{\pi} dy \|u^*\|^2_{H^r_w(-1,1)}. \tag{3.14}$$

Now (3.12) is a consequence of (3.1), (3.6), and (3.13), (3.14). ∎

Remark 3.1. As it can be easily checked, the previous proof allows one to get (3.12) also for the case where $\Omega=(-1,1)^m \times (-\pi,\pi)^n$, $m, n \geqslant 1$, and Chebyshev interpolation is used in $(-1,1)^m$ while Fourier interpolation is used in $(-\pi,\pi)^n$. In this case it should be assumed that $mr^{-1}+ns^{-1}<2$, so that every $u \in H^{r,s}(\Omega)$ is continuous in $\bar{\Omega}$.

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